

MAXIMAL CROSSED PRODUCT ORDERS OVER DISCRETE VALUATION RINGS

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ABSTRACT. The problem of determining when a (classical) crossed product $T = S^f * G$ of a finite group G over a discrete valuation ring S is a maximal order, was answered in the 1960's for the case where S is tamely ramified over the subring of invariants S^G . The answer was given in terms of the conductor subgroup (with respect to f) of the inertia. In this paper we solve this problem in general when S/S^G is residually separable. We show that the maximal order property entails a restrictive structure on the sub-crossed product graded by the inertia subgroup. In particular, the inertia is abelian. Using this structure, one is able to extend the notion of the conductor. As in the tame case, the order of the conductor is equal to the number of maximal two sided ideals of T and hence to the number of maximal orders containing T in its quotient ring. Consequently, T is a maximal order if and only if the conductor subgroup is trivial.

1. INTRODUCTION

Let S be a discrete valuation ring (DVR) and let G be a finite subgroup of $\text{Aut}(S)$. Denote the unique maximal ideal of S by M_S and the corresponding residue field S/M_S by \bar{S} . For any $f \in Z^2(G, S^*)$ consider the crossed product $T := S^f * G = \bigoplus_{g \in G} S U_g$ with multiplication

$$(1.1) \quad s U_g t U_h = s g(t) f(g, h) U_{gh} \quad s, t \in S, \quad g, h \in G.$$

Let $R := S^G$ be the subring of G -invariant elements in S and let $\bar{R} := R/(M_S \cap R)$ be its residue field. We shall always assume that the extension \bar{S}/\bar{R} is separable (*residual separability* property of S/R). Denote the field of quotients of S by L . Then the 2-cocycle f can be regarded also as in $Z^2(G, L^*)$, and T is an R -order in the central simple algebra $L^f * G$.

Question 1.1. When is the R -order T maximal in $L^f * G$?

Suppose that S/R is *tamely ramified*, that is when the order of the inertia subgroup $G_I \triangleleft G$ is prime to $p := \text{char}(\bar{R})$. In this case the answer to Question 1.1 can be given in terms of the subgroup $H_f \triangleleft G$, which is maximal in the inertia subgroup such that the cohomology class $[f] \in H^2(G, \bar{S}^*)$ is inflated from $H^2(G/H_f, \bar{S}^*)$, namely the *conductor* subgroup with respect to f .

Theorem 1.2. [12, Theorem 2.5] *Let S/R be a tamely ramified extension. Then the number of maximal R -orders containing T in $L^f * G$ is equal to the order of the conductor H_f . In particular, T is a maximal R -order if and only if H_f is trivial.*

Question 1.1 is discussed in [8] in a special instance of the tamely ramified case, namely where L is a finite extension of the p -adic rationals \mathbb{Q}_p . The number of maximal R -orders containing T in $L^f * G$ is given there in terms of the Schur index of the class $[f] \in H^2(G, L^*)$.

These results are generalized in [5, 11] for any extension S/R such that the residue fields are finite. However, under this condition on the residue fields, T cannot be a maximal R -order unless S/R is again tamely ramified (by [7, Theorem 2] and Theorem 2.1 hereafter).

In this note we answer Question 1.1 dropping the above tame ramification assumption. We first show

Theorem A. *If $S^f * G$ is a maximal order or, more generally, a hereditary R -order, then the inertia subgroup of G is abelian.*

With the restrictive structure that the heredity property entails on T (Corollary 2.4), we are able to extend the notion of the conductor subgroup (Definition 3.1). This notion arises naturally from a well known group cohomology map. It turns out that the image of this map controls the number of maximal two sided ideals of T or, equivalently, the number of maximal orders in $L^f * G$ which contain T (Corollary 3.11). This implies that as in the tamely ramified case, the maximal order property of T depends on the triviality of the conductor:

Theorem B. *Let $T = S^f * G$ be a hereditary crossed product order. Then the number of maximal R -orders containing T in $L^f * G$ is equal to the order of the conductor H_f . In particular, T is a maximal R -order if and only if it is hereditary and the conductor H_f is trivial.*

Finally, the demand that S is a DVR can be relaxed to the more general case where S is a Dedekind domain, $R = S^G$ is a DVR and S/R is residually separable. The reduction is fairly standard and appears in Section 4.

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2. HEREDITY AND SEMISIMPLICITY

The following result will be useful in the sequel.

Theorem 2.1. [1, Theorem 2.3] *Let R be a DVR and let Λ be an R -order. Then Λ is maximal if and only if it is hereditary and has a unique two sided ideal.*

We first handle the heredity condition in Theorem 2.1. In order to formulate the criterion, note that $T/M_S T$ is isomorphic to the crossed product $\bar{S}^{\bar{f}} * G$. The action of G on \bar{S} is induced by its action on S and hence admits a kernel. This kernel is the inertia (or the first ramification) subgroup G_I . The 2-cocycle \bar{f} is the image of f under the natural map $Z^2(G, S^*) \rightarrow Z^2(G, \bar{S}^*)$. We have

Theorem 2.2. [4, Theorem A] *With the above notation, $S^f * G$ is a hereditary order if and only if $\bar{S}^{\bar{f}} * G$ is semi-simple.*

If S/R is tamely ramified, then the fact that the order of G_I is invertible in the field \bar{S} implies that $\bar{S}^{\bar{f}} * G$ is semi-simple independently of \bar{f} , by a generalized Maschke's Theorem. Hence T is hereditary. However, it turns out that T may be hereditary even when S/R is not tamely ramified [4, Example 4.1].

Here is an explicit criterion for the semisimplicity of $\bar{S}^{\bar{f}} * G$. By Theorem 2.2, it is a necessary and sufficient condition for the heredity property of $S^f * G$. Note that since the inertia subgroup $G_I \triangleleft G$ acts trivially on \bar{S} , the sub-crossed product graded by G_I is a twisted group algebra $\bar{S}^{\bar{f}} G_I$. Let P be a p -Sylow subgroup of G_I , where p is the characteristic of the residue field \bar{S} (in case $p = 0$, take P as the trivial group).

Theorem 2.3. [2, Theorem 2], *With the above notation, $\bar{S}^{\bar{f}} * G$ is semi-simple if and only if the twisted group subalgebra $F := \bar{S}^{\bar{f}} P$ is a purely inseparable field extension of \bar{S} . In particular, P is abelian and the 2-cocycle \bar{f} is non-trivial on any non-trivial subgroup of P . Additionally, it follows that the order of the commutator subgroup $[G_I, G_I]$ is prime to p .*

Proof of Theorem A. Let w be a generator of M_S . For any $\sigma \in G_I$ let $\sigma(w) = x_\sigma w$, where $x_\sigma \in S^*$. Then by [13, Theorem 25, P. 295], the map $\sigma \mapsto \bar{x}_\sigma$ is a homomorphism from G_I into \bar{S}^* whose kernel under the residual separability assumption is exactly P (the second ramification group). Consequently, P is normal and $G_I = P \rtimes C_{e_0}$, where $C_{e_0} = \langle \sigma_0 \rangle$ is a cyclic group whose order is prime to p . Now, if $\bar{S}^{\bar{f}} * G$ is semisimple, then by Theorem 2.3, the order of the commutator $[G_I, G_I]$ is prime to p , and hence the action of C_{e_0} on P is trivial. Consequently, G_I is a direct product of P and C_{e_0} , hence abelian. \square

The following is a stronger consequence of the semisimplicity of $\bar{S}^{\bar{f}} * G$. For the sake of convenience, we continue to denote the basis elements of $\bar{S}^{\bar{f}} * G$ by $\{U_\sigma\}_{\sigma \in G}$. For any $\sigma \in P$, let $\lambda := U_\sigma U_{\sigma_0} U_\sigma^{-1} U_{\sigma_0}^{-1} \in \bar{S}^*$. Suppose that the order of σ is p^m for some m . Then $\lambda^{p^m} := U_\sigma^{p^m} U_{\sigma_0} U_\sigma^{-p^m} U_{\sigma_0}^{-1} = 1$. Since \bar{S} does not admit non-trivial p -th roots of 1, we deduce that $\lambda = 1$ and thus U_{σ_0} is central in $\bar{S}^{\bar{f}} G_I$. Let $\alpha_0 := U_{\sigma_0}^{e_0} \in \bar{S}^*$. We obtain

Corollary 2.4. *T is hereditary if and only if $\bar{S}^{\bar{f}} G_I$ is semisimple and isomorphic to a commutative twisted group algebra $F^{\alpha_0} C_{e_0} \simeq F[x]/\langle x^{e_0} - \alpha_0 \rangle$, where $F = \bar{S}^{\bar{f}} P$ is a purely inseparable extension of \bar{S} and C_{e_0} is a cyclic group of order e_0 , which is prime to p . In particular, G_I is abelian of the form $G_I = P \times C_{e_0}$.*

3. THE NUMBER OF SIMPLE COMPONENTS OF $\bar{S}^{\bar{f}} * G$

Suppose that T is hereditary. Then by Theorems 2.2 and 2.3, the restriction of \bar{f} to a subgroup H of G_I can be trivial only if H is a p' -group, that is H is contained in C_{e_0} and hence normal in G . Due to this observation, one can generalize the definition of the conductor subgroup as follows.

Definition 3.1. Let $f \in Z^2(G, S^*)$ such that $S^f * G$ is hereditary. The conductor H_f with respect to f is the maximal subgroup of the inertia such that the class $[f] \in H^2(G, S^*)$ is inflated from $H^2(G/H_f, S^*)$.

In subsection 3.2 we make use of the above definition so as to obtain that the number of simple components of $\bar{S}^{\bar{f}} * G$ and the number of maximal two-sided ideals in T are both equal to the order of the conductor subgroup H_f (compare with [12,

Theorem 2.5]). By that Theorem B will be deduced, since the number of maximal R -orders containing T in $L^f * G$ is equal to the number of maximal two-sided ideals in T [6, Theorem 1.7].

The proof of Theorem B is partially based on [12]. Subsection 3.1 below proposes a cohomological interpretation to this result.

3.1. In this subsection we present the cohomological tool for the calculation of the number of simple components of $\tilde{S}^f * G$ that is essential for Theorem B. The discussion is based on a construction due to J.P. Serre and can be found in [9, Section 1.7]. Here is a brief description.

Let

$$(3.1) \quad 1 \rightarrow A \rightarrow G \rightarrow G/A \rightarrow 1$$

be an extension of finite groups, where A is abelian. As usual, G/A acts on A via the conjugation in G , namely, for every $\bar{g} \in G/A$ and $a \in A$, $\bar{g}(a) = gag^{-1}$. Next, let M be a G -module which is A -trivial, that is G acts on M via G/A . Then the action of G/A on A induces the following diagonal action of G/A on $\text{hom}(A, M) (\simeq H^1(A, M))$. Let $\bar{g} \in G/A$ and $\varphi \in \text{hom}(A, M)$. Then $\bar{g}(\varphi) \in \text{hom}(A, M)$ is defined on $a \in A$ via the pairing

$$(3.2) \quad \langle \bar{g}(\varphi), a \rangle = \bar{g} \langle \varphi, \bar{g}^{-1}(a) \rangle.$$

Next, let $f \in Z^2(G, M)$ satisfy

$$(3.3) \quad f(g_1, g_2) = f(g_1, g_2a), \quad \forall g_1, g_2 \in G, a \in A.$$

In particular, the restriction of f to A is trivial.

For any $a \in A$ and $\bar{g} \in G/A$ define

$$(3.4) \quad \begin{array}{ccc} \pi_f(\bar{g}) : A & \rightarrow & M \\ a & \mapsto & f(a, g). \end{array}$$

Theorem 3.2. (see [9, Theorem 7.3, P. 60]) Let $\text{res}_A^G : H^2(G, M) \rightarrow H^2(A, M)$ and $\text{inf}_G^{G/A} : H^2(G/A, M) \rightarrow H^2(G, M)$ be the restriction and inflation maps respectively. Let π_f be as in (3.4). Then

- (1) Any class in $\ker(\text{res}_A^G)$ admits a representative $f \in Z^2(G, M)$ satisfying (3.3).
- (2) For any $\bar{g} \in G/A$, $\pi_f(\bar{g}) \in \text{hom}(A, M)$.
- (3) $\pi_f(\bar{g})$ does not depend on the choice of the representative $g \in G$ for \bar{g} .
- (4) The map $\bar{g} \mapsto \pi_f(\bar{g})$ is a 1-cocycle from G/A to $\text{hom}(A, M)$.
- (5) If $f' \in [f]$ satisfies (3.3), then the 1-cocycles $\pi_{f'}$ and π_f differ by a 1-coboundary.
- (6) If f_1 and f_2 satisfy (3.3), then so does $f_1 + f_2$. Moreover, $\pi_{f_1 + f_2} = \pi_{f_1} + \pi_{f_2}$.
- (7) $\pi_f \in B^1(G/A, \text{hom}(A, M))$ if and only if the cohomology class $[f]$ is in the image of $\text{inf}_G^{G/A}$.

Corollary 3.3. (see [9, Theorem 7.3, P. 60]) The map $\Pi : [f] \bmod [\text{im}(\text{inf}_G^{G/A})] \mapsto [\pi_f]$ is a well defined injection of $\ker(\text{res}_A^G)/\text{im}(\text{inf}_G^{G/A})$ into $H^1(G/A, \text{hom}(A, M))$.

The map Π is applied for crossed products as follows. Let $K^f * G = \oplus_{g \in G} KU_g$ be a crossed product, where K is a field and $f \in Z^2(G, K^*)$. Suppose that $A \triangleleft G$ is an abelian subgroup acting trivially on K such that the restriction of f to A is

cohomologically trivial. By Theorem 3.2(1), the K -basis $\{U_g\}_{g \in G}$ may be chosen such that

$$(3.5) \quad U_{ga} = U_g U_a, \forall g \in G, a \in A.$$

In particular, $K^f * G$ contains the ordinary group algebra KA . Then G/A acts on KA via the conjugation in $K^f * G$. We describe this action using the 1-cocycle $\pi_f \in Z^1(G/A, \text{hom}(A, K^*))$. Let $\bar{g} \in G/A$ and $a \in A$. Then by (3.4),

$$(3.6) \quad \langle \pi_f(\bar{g}), a \rangle = f(a, g) = U_a U_g U_{ag}^{-1} = U_a U_g U_{g\bar{g}^{-1}(a)}^{-1}.$$

By (3.5), $U_{g\bar{g}^{-1}(a)}^{-1} = U_{\bar{g}^{-1}(a)}^{-1} U_g^{-1}$. Consequently,

$$(3.7) \quad \langle \pi_f(\bar{g}), a \rangle = U_a U_g U_{\bar{g}^{-1}(a)}^{-1} U_g^{-1}.$$

Hence, for every $\bar{g} \in G/A$ and $a \in A$

$$(3.8) \quad \bar{g}(U_a) = U_g U_a U_g^{-1} = (U_g U_{\bar{g}^{-1}(\bar{g}(a))}^{-1} U_g^{-1})^{-1} = \langle \pi_f(\bar{g}), \bar{g}(a) \rangle^{-1} U_{\bar{g}(a)}.$$

Now, suppose that $|A|$ is invertible in K . Then the primitive idempotents of KA are $\iota_\chi = \frac{1}{|A|} \sum_{a \in A} \langle \chi, a \rangle^{-1} U_a$ for every $\chi \in \text{hom}(A, K^*)$. The action on KA yields an action of G/A on the set of primitive idempotents of KA as follows.

Proposition 3.4. *(see a special instance in [3, Proposition 2.9]) With the above notation, let $\bar{g} \in G/A$ and let $\chi \in \text{hom}(A, K^*)$. Then $\bar{g}(\iota_\chi) = \iota_{\bar{g}(\chi)} \pi_f(\bar{g})$.*

Proof.

$$\bar{g}(\iota_\chi) = U_g \iota_\chi U_g^{-1} = U_g \frac{1}{|A|} \sum_{a \in A} \langle \chi, a \rangle^{-1} U_a U_g^{-1} = \frac{1}{|A|} \sum_{a \in A} \bar{g} \langle \chi, a \rangle^{-1} U_g U_a U_g^{-1}.$$

Then by (3.8),

$$\begin{aligned} \bar{g}(\iota_\chi) &= \frac{1}{|A|} \sum_{a \in A} \bar{g} \langle \chi, a \rangle^{-1} \langle \pi_f(\bar{g}), \bar{g}(a) \rangle^{-1} U_{\bar{g}(a)} \\ &= \frac{1}{|A|} \sum_{a \in A} \langle \bar{g}(\chi), \bar{g}(a) \rangle^{-1} \langle \pi_f(\bar{g}), \bar{g}(a) \rangle^{-1} U_{\bar{g}(a)} \\ &= \frac{1}{|A|} \sum_{a \in A} \langle \bar{g}(\chi) \pi_f(\bar{g}), a \rangle^{-1} U_a = \iota_{\bar{g}(\chi) \pi_f(\bar{g})}. \end{aligned}$$

□

3.2. The second step in determining if T is a maximal order, after having taken care of its heredity property (in Section 2), is to handle the locality condition in Theorem 2.1. We have

Proposition 3.5. *The number of maximal two-sided ideals in T is equal to the number of maximal two-sided ideals in $\bar{S}^f * G$. In particular, T is local if and only if so is $\bar{S}^f * G$.*

Proof. This is clear since every maximal two sided ideal of T contains $M_S T$. □

Assume that $\bar{S}^f * G = \text{Span}_{\bar{S}} \{U_g\}_{g \in G}$ satisfies the necessary and sufficient condition for semisimplicity in Corollary 2.4. Then by Proposition 3.5, the number of maximal two-sided ideals in T is equal to the number of simple components of $\bar{S}^f * G$. In particular, by Theorem 2.1, T is a maximal order if and only if $\bar{S}^f * G$ admits a single simple component.

We need to deal with the following

Question 3.6. Let $\bar{S}^{\bar{f}} * G = T/M_S T$ be a crossed product as above. Suppose that $\bar{S}^{\bar{f}} * G$ is semisimple. How many simple components does $\bar{S}^{\bar{f}} * G$ admit? In particular, when is $\bar{S}^{\bar{f}} * G$ simple?

In general, determining the number of simple components of an arbitrary semi-simple crossed product $K^f * G$ of a finite group G over a field K might be hard. Suppose that $[f] \in \ker(\text{res}_A^G)$ for an abelian subgroup $A \triangleleft G$ which acts trivially on K (and by Theorem 3.2(1) we may assume that f satisfies (3.3)). Then a necessary condition for the simplicity of $K^f * G$ is that the primitive idempotents of the commutative group ring KA belong to the same orbit under the action of G . By Proposition 3.4, this implies that the 1-cocycle π_f is *onto* $\text{hom}(A, K^*)$. Under our conditions however, the central idempotents of $\bar{S}^{\bar{f}} * G$ can be calculated using Proposition 3.4, as well as the structure of $\bar{S}^{\bar{f}} * G_I$ given in Corollary 2.4.

The following claim shows that the central primitive idempotents of $\bar{S}^{\bar{f}} * G$ are supported by the inertia subgroup.

Proposition 3.7. *The center of $\bar{S}^{\bar{f}} * G$ lies in $\bar{S}^{\bar{f}} G_I$.*

Proof. Let $y = \sum_{g \in G} \bar{s}_g U_g \in \bar{S}^{\bar{f}} * G$. Suppose that $\bar{s}_{g_0} \neq 0$ for some $g_0 \notin G_I$. Then since g_0 is not in the kernel of the action of G on \bar{S} , there exists an element $\bar{t} \in \bar{S}$ which does not commute with U_{g_0} and hence also with y . \square

In view of Proposition 3.7, any central idempotent of $\bar{S}^{\bar{f}} * G$ is a sum of certain primitive idempotents of the commutative twisted group subalgebra $\bar{S}^{\bar{f}} G_I$. By Corollary 2.4, $\bar{S}^{\bar{f}} G_I$ is isomorphic to the commutative twisted group ring $F^{\alpha_0} C_{e_0} = \text{Span}_F \{U_{\sigma_0^i}\}_{0 \leq i \leq e_0-1}$, where $(U_{\sigma_0})^{e_0} = \alpha_0 \in \bar{S}^*$. We need the following properties of the field \bar{S} .

Proposition 3.8. *With the above notation*

- (1) *The field \bar{S} contains all e_0 -th roots of 1.*
- (2) *Let $\zeta_{e_0} \in \bar{S}$ be an e_0 -th root of 1. Then for every $g \in G$, $g\sigma_0 g^{-1} = \sigma_0^m$ where m is determined by $g(\zeta_{e_0}) = \zeta_{e_0}^m$.*

Proof. The map $\sigma \mapsto \bar{x}_\sigma$ in the proof of Theorem A yields an embedding of the cyclic group C_{e_0} in \bar{S}^* verifying (1). In order to prove (2), we need to show that this map is also a G -morphism. As can easily be seen, the map does not depend on the generator w of M_S . Choosing $g^{-1}(w)$ as a new generator we obtain that $\sigma(g^{-1}(w)) = y_\sigma g^{-1}(w)$, where $\bar{y}_\sigma = \bar{x}_\sigma$. Acting with g on both sides gives $g\sigma g^{-1}(w) = g(y_\sigma)w$. Hence $\bar{x}_{g\sigma g^{-1}} = g(\bar{x}_\sigma)$ and we are done. \square

Let Γ_f be a maximal subgroup of G_I such that the restriction of \bar{f} to it is cohomologically trivial (compare with [12, P. 111, Definition]). By Theorem 2.3, Γ_f intersects P trivially, hence it is contained in $C_{e_0} = \langle \sigma_0 \rangle$ and therefore it is unique. Let $c = c(f)$ be such that $\Gamma_f := \langle \sigma_0^c \rangle$. Then the order of Γ_f is $d = d(f) = \frac{e_0}{c}$, which is the maximal divisor of e_0 such that α_0 admits a root of order d in \bar{S} (equivalently in F , since \bar{S} is its separable closure inside F and $(e_0, p) = 1$). In particular, d is invertible in \bar{S} .

It is clear that every subgroup A of Γ_f is normal in G , since these subgroups are contained in the cyclic normal subgroup C_{e_0} . The map $\pi_{\bar{f}}$ may therefore be

applied for every $A \triangleleft \Gamma_f$ and $M := \bar{S}^*$. By Theorem 3.2(1), putting $A := \Gamma_f$, we may assume that \bar{f} satisfies (3.3). In particular, $U_{\sigma_0^{cj}} = U_{\sigma_0^c}^j$ for every integer j . Now, by Proposition 3.8(1), \bar{S} contains a primitive d -th roots of 1, denoted by ζ_d . Let k be a divisor of d , and let $A := \langle \sigma_0^{kc} \rangle$ be a subgroup of Γ_f of order $\frac{d}{k}$. Then $\text{hom}(A, M) = \text{hom}(A, \bar{S}^*)$ is a cyclic group of order $\frac{d}{k}$ whose elements are determined by the generator σ_0^{kc} as follows.

$$(3.9) \quad \chi_j^{(k)} : \sigma_0^{kc} \mapsto \zeta_d^{kj}, \quad 0 \leq j \leq \frac{d}{k} - 1.$$

The idempotents of $\bar{S}^{\bar{f}} G_I$ can now be given explicitly. For $A = \Gamma_f$ put $k = 1$ and let $\chi_j = \chi_j^{(1)}$ in (3.9).

Proposition 3.9. *The elements*

$$(3.10) \quad \iota_j = \frac{1}{d} \sum_{l=0}^{d-1} \langle \chi_j, \sigma_0^{cl} \rangle^{-1} U_{\sigma_0^{cl}}, \quad 0 \leq j \leq d-1$$

form a complete set of primitive orthogonal idempotents of $\bar{S}^{\bar{f}} G_I$.

Proof. Since $\bar{S}^{\bar{f}} G_I \simeq F^{\alpha_0} C_{e_0}$, one can apply [12, Proposition 2.2] putting F as the base field. \square

The number of simple components of $\bar{S}^{\bar{f}} G$ depends on the action of G/Γ_f on the above idempotents. We have

Proposition 3.10. *For every $A = \langle \sigma_0^{kc} \rangle \triangleleft \Gamma_f$, the action of G/A on $\text{hom}(A, \bar{S}^*)$ is trivial.*

Proof. Let $\bar{g} \in G/A$ and suppose that $\bar{g}^{-1}(\sigma_0) = g^{-1}\sigma_0 g = \sigma_0^m$. Then by Proposition 3.8(2), $\bar{g}^{-1}(\zeta_d^k) = g^{-1}(\zeta_d^k) = \zeta_d^{km}$. Now, let $\chi_j^{(k)} \in \text{hom}(A, \bar{S}^*)$. Then $\langle \bar{g}(\chi_j^{(k)}), \sigma_0^{kc} \rangle = \bar{g} \langle \chi_j^{(k)}, \bar{g}^{-1}(\sigma_0^{kc}) \rangle = \bar{g} \langle \chi_j^{(k)}, \sigma_0^{kcm} \rangle = \bar{g}(\zeta_d^{kmj}) = \zeta_d^{kj} = \langle \chi_j^{(k)}, \sigma_0^{kc} \rangle$, proving that $\bar{g}(\chi_j^{(k)}) = \chi_j^{(k)}$. \square

By Propositions 3.4 and 3.10 for $A = \Gamma_f$, we obtain that an element $\bar{g} \in G/\Gamma_f$ acts on the idempotents of $\bar{S}^{\bar{f}} G_I$ as translations by $\pi_{\bar{f}}(\bar{g})$. More precisely

$$(3.11) \quad \bar{g}(\iota_{\chi_j}) = \iota_{\chi_j \pi_{\bar{f}}(\bar{g})}, \quad \bar{g} \in G/\Gamma_f, \quad 0 \leq j \leq d-1.$$

By Proposition 3.10, the 1-cocycle $\pi_{\bar{f}} : G/\Gamma_f \rightarrow \text{hom}(\Gamma_f, \bar{S}^*)$ is in fact a group homomorphism. By (3.11), there is a 1-1 correspondence between the orbits induced by the action of G/Γ_f on the set of primitive idempotents of $\bar{S}^{\bar{f}} G_I$ and the cosets of the image $\pi_{\bar{f}}(G/\Gamma_f)$ in $\text{hom}(\Gamma_f, \bar{S}^*)$ (and hence all the orbits are of the same cardinality).

Next, by Propositions 3.7 and 3.9, any central idempotent of $\bar{S}^{\bar{f}} * G$ is of the form $\iota = \sum_{j \in B} \iota_j$, where $B \subset \{0, \dots, d-1\}$ is a set of indices of an orbit of primitive idempotents of $\bar{S}^{\bar{f}} G_I$ under the action of G/Γ_f .

Here is an answer to Question 3.6 in terms of the image of $\pi_{\bar{f}}$.

Corollary 3.11. *Let $T = S^f * G$ be a hereditary crossed product. Then the number of simple components of $T/M_S T = \bar{S}^{\bar{f}} * G$ is equal to the index of $\pi_{\bar{f}}(G/\Gamma_f)$ in $\text{hom}(\Gamma_f, \bar{S}^*)$. In particular, $\bar{S}^{\bar{f}} * G$ is simple if and only if $|\pi_{\bar{f}}(G/\Gamma_f)| = |\text{hom}(\Gamma_f, \bar{S}^*)| = d$.*

We now show that the number of simple components of $\bar{S}^f * G$, which is the same as the number of maximal orders containing T in $L^f * G$, is equal to the order of the conductor.

Proof of Theorem B. Let I_f be the index of $\pi_{\bar{f}}(G/\Gamma_f)$ in $\text{hom}(\Gamma_f, \bar{S}^*)$ as above. Then I_f is the order of the maximal subgroup $A \triangleleft \Gamma_f$ such that the decomposition

$$(3.12) \quad G/\Gamma_f \xrightarrow{\pi_{\bar{f}}} \text{hom}(\Gamma_f, \bar{S}^*) \xrightarrow{\text{res}} \text{hom}(A, \bar{S}^*)$$

is trivial, where the right map is the restriction map from Γ_f to A . Now, consider the diagram

$$(3.13) \quad \begin{array}{ccc} G/A & \longrightarrow & G/\Gamma_f \\ \downarrow & & \downarrow \\ \text{hom}(A, \bar{S}^*) & \xleftarrow{\text{res}} & \text{hom}(\Gamma_f, \bar{S}^*) \end{array},$$

where the vertical arrows stand for the maps $\pi_{\bar{f}}$ with respect to the normal subgroups A and Γ_f and the upper horizontal map is the natural projection. By the definition of $\pi_{\bar{f}}$ (3.4) and Theorem 3.2(3), we deduce that the diagram (3.13) is commutative. Consequently, A is the maximal subgroup of Γ_f such that the map $G/A \xrightarrow{\pi_{\bar{f}}} \text{hom}(A, \bar{S}^*)$ is trivial. Equivalently, since the action of G/A on $\text{hom}(A, \bar{S}^*)$ is trivial (Proposition 3.10), A is the maximal subgroup of Γ_f such that $\pi_{\bar{f}}$ with respect to A is a 1-coboundary. By Definition 3.1 and Theorem 3.2(7), we obtain that A coincides with the conductor H_f . Applying Corollary 3.11, we obtain that the number of simple components of $\bar{S}^f * G$ is equal to the order of H_f . By Proposition 3.5, $|H_f|$ is the number of maximal two-sided ideals in T and by [6, Theorem 1.7], it is the number of maximal R -orders containing T in $L^f * G$. This completes the proof of Theorem B. \square

4. A REDUCTION ARGUMENT

In this section we reduce the question of when $T = S^f * G$ is a maximal order from the case where S is a Dedekind domain and R is a DVR (keeping the demand that S/R is residually separable) to the case discussed in the previous sections, namely where both S and R are DVR's.

Let z generate the unique maximal ideal of R and let $\hat{R} := \varprojlim_i R/z^i R$ and $\hat{S} := \varprojlim_i S/z^i S$ be the corresponding completions. The action on S determines an action of G also on \hat{S} . Denote the primitive idempotents of \hat{S} by e_1, \dots, e_k . For every $1 \leq j \leq k$, let $G_j := \{g \in G | g(e_j) = e_j\}$ be the decomposition group which corresponds to the primitive idempotent e_j . Let $\hat{T} := \varprojlim_i T/z^i T$. Then $\hat{T} \simeq \hat{S} * G$, where the action and 2-cocycle in the crossed product of G over \hat{S} are induced from those in T . The crossed product \hat{T} is an \hat{R} -order. For every $1 \leq j \leq k$, let $T_j := e_j \hat{T} e_j$. Then by the definition of the decomposition groups, $T_j = \hat{S} e_j * G_j$. Since $\hat{S} e_j$ is a DVR, we know how to determine if T_j is a maximal \hat{R} -order. The reduction is established by passing from T to T_1 by the following

Theorem 4.1. *The following are equivalent*

- (1) T is a maximal R -order.
- (2) \hat{T} is a maximal \hat{R} -order.
- (3) T_1 is a maximal \hat{R} -order.

Furthermore, the number of two sided maximal ideals of the above three algebras is equal.

Proof. (1) \Leftrightarrow (2) The number of maximal two sided ideals of T does not change when passing to the completion \widehat{T} . Now, by Theorem 2.1, it remains to show that T is hereditary if and only if so is \widehat{T} . Indeed, since $S/\text{Jac}(S)$ and $\widehat{S}/\text{Jac}(\widehat{S})$ are both isomorphic to k copies of the residue field of S (where Jac denotes the Jacobson radical), we obtain that $\bar{T} := T/\text{Jac}(S)T \simeq \widehat{T}/\text{Jac}(\widehat{S})\widehat{T}$. By [4, Theorem A] (a general version of Theorem 2.2), both T and \widehat{T} are hereditary if and only if \bar{T} is semisimple.

(2) \Leftrightarrow (3) Since the action of G on the set $\{e_j\}_{j=1}^k$ of primitive idempotents of \widehat{S} is transitive, it follows that for every $1 \leq j \leq k$, $e_j \in \widehat{T}e_1\widehat{T}$. Consequently, $1 \in \widehat{T}e_1\widehat{T}$ and hence $\widehat{T} = \widehat{T}e_1\widehat{T}$. By [10, Proposition 3.5.6], we deduce that \widehat{T} and $e_1\widehat{T}e_1$ are Morita equivalent and we are done. \square

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